Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00101-0

J. Appl. Maths Mechs, Vol. 65, No. 6, pp. 951–961, 2001

0021-8928/01/$—see front matter

SOME INVERSE PROBLEMS FOR A VISCOELASTIC MEDIUM WITH A PHYSICALLY NON-LINEAR INCLUSION†

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(*Received* 20 *September* 2000)

A plane finite viscoelastic domain with a physically non-linear inclusion of arbitrary form is considered. The problem of finding those loads which, acting on the outer boundary of the domain, are such that they produce a specified uniform stress–strain state in the inclusion, is solved. Examples, in particular, of the optimal deformation and fracture of the inclusion under creep conditions, are considered.

A plane finite elastic domain with a physically non-linear inclusion of arbitrary form, in which it is necessary to produce the required uniform stress–strain state by choosing appropriate external forces, was considered earlier in [1]. In this paper the problem is extended to the case of a linear viscoelastic domain with a physically non-linear inclusion. An inclusion which weakens during creep is considered separately, and inverse problems relating to it concerning the choice of external forces leading to the optimal (in the sense indicated below) paths for the deformation of the inclusion and its fracture as a consequence of creep are investigated.

1. THE STRESS–STRAIN STATE OF A VISCOELASTIC DOMAIN CONTAINING A PHYSICALLY NON-LINEAR INCLUSION WITH A SPECIFIED STRESS–STRAIN STATE

Consider a finite, isotropic viscoelastic domain S of the Ox_1x_2 plane with a physically non-linear inclusion S^*. The outer and inner boundaries of the domain S are the simple closed contours L and L^* (that is, L^* separates S from S^*).

Extending Hooke's law for the plane problem in [1] to the case of linear viscoelasticity by replacing the constants of elasticity by the corresponding operators [2], in the domain S, we will have

$$8\tilde{\mu}\varepsilon_{kl} = (\tilde{\varkappa}-1)\sigma_{nn}\delta_{kl} + 4\sigma^0_{kl}, \quad k, l = 1, 2 \tag{1.1}$$
$$\sigma^0_{kl} = \sigma_{kl} - \sigma_{nn}\delta_{kl}/2, \quad \tilde{\mu} = \mu(1+\tilde{K}_1), \quad \tilde{\varkappa} = \varkappa(1+\tilde{K}_2)$$

where μ and $\varkappa$ are constants of elasticity, $\tilde{K}_1$ and $\tilde{K}_2$ are Volterra operators and the remaining notation is the same as that used previously in [1]; summation from 1 to 2 is carried out over repeated subscripts. The system of coordinates is chosen so that the point $(0, 0) \in S^*$.

We select the constitutive equations for the inclusion S^* in the form [1]

$$\varepsilon^*_{kl} = \tilde{F}_{kl}(\sigma^*_{mn}), \quad k, l, m, n = 1, 2 \tag{1.2}$$

where $\tilde{F}_{kl}$ are non-linear operators of fairly general form.

We will now formulate the inverse problem, which generalizes the problem considered earlier in [1]: on the boundary L, it is required to select those loads which will produce, in the physically non-linear inclusion S^*, the required uniform stress–strain state characterized by the stresses $\sigma^*_{kl} = \sigma^*_{kl}(t)$ and strains $\varepsilon^*_{kl} = \varepsilon^*_{kl}(t)$ $(k, l = 1, 2)$. These stresses and strains are related by Eqs (1.2). At the initial instant of time $t = 0$, the whole of the domain $S^* \cup S$ is in the natural undeformed state. On the boundary L^*, the load field p_k and the displacement field u_k $(k = 1, 2)$ are continuous. The problem is consider in a geometrically linear formulation.

†*Prikl. Mat. Mekh.* Vol. 65, No. 6, pp. 983–994, 2001.

Since σ^*_{kl} and ε^*_{kl} are independent of the coordinates x_1 and x_2, arguments similar to those presented earlier in [1] for the stress function $U^* = U^*(z, \bar{z}, t)$ and the complex displacement function $w^* = u^*_1 + iu^*_2$ (under the assumption that $w^*(0, 0, t) = 0$) in the inclusion S^* give

$$2U^* = Az\bar{z} + Bz^2/2 + \bar{B}\bar{z}^2/2$$
$$2A(t) = \sigma^*_{11} + \sigma^*_{22}, \quad 2B(t) = \sigma^*_{22} - \sigma^*_{11} + 2i\sigma^*_{12} \tag{1.3}$$

$$2w^* = Cz + D\bar{z}$$
$$C(t) = \varepsilon^*_{11} + \varepsilon^*_{22} + 2i\varepsilon^*, \quad D(t) = \varepsilon^*_{11} - \varepsilon^*_{22} + 2i\varepsilon^*_{12} \tag{1.4}$$

(ε^* is the magnitude of the "rotation" in the domain S^*).

We will denote the function which conformally maps the infinite domain located outside L^* onto the exterior of the unit circle γ^* of the complex plane ζ by $z = \omega(\zeta)$ ($\zeta = \rho e^{i\theta}$). Then, by virtue of the continuity of the loads and the displacements on the boundary L^*, from (1.3) and (1.4), by analogy with the procedure described earlier in [1] involving, in accordance with Volterra's principle [2], the replacement of the constants of elasticity μ and $\varkappa$ by the viscoelastic operators $\tilde{\mu}$ and $\tilde{\varkappa}$ of the form (1.1), we shall have, when $|\zeta| = 1$, for the functions $\varphi(\zeta, t)$ and $\psi(\zeta, t)$ defining the stress–strain state in the domain S

$$\overline{\varphi(\sigma, t)} + \overline{\omega(\sigma)}\varphi'(\sigma, t)/\omega'(\sigma) + \psi(\sigma, t) = A(t)\overline{\omega(\sigma)} + B(t)\omega(\sigma)$$
$$\tilde{\varkappa}\overline{\varphi(\sigma, t)} - \overline{\omega(\sigma, t)}\varphi'(\sigma, t)/\omega'(\sigma) - \psi(\sigma, t) = \tilde{\mu}[\overline{C(t)}\ \overline{\omega(\sigma)} + \overline{D(t)}\omega(\sigma)] \tag{1.5}$$

on γ^*; $\sigma = e^{i\theta}$.

The boundary condition for the function $\varphi(\zeta, t)$ follows from relations (1.5) and, from the boundary condition for this function, we obtain the Volterra integral equation of the second kind

$$(\tilde{\varkappa} + 1)\varphi(\zeta, t) = [A(t) + \tilde{\mu}C(t)]\omega(\zeta) + [\overline{B(t)} + \tilde{\mu}D(t)]\bar{\omega}(\zeta^{-1}), \quad |\zeta| > 1 \tag{1.6}$$

If the function $\varphi(\zeta, t)$ has been found, then for $\psi(\zeta, t)$, as a consequence of the first boundary condition of (1.5), we will have [1]

$$\psi(\zeta, t) = -\bar{\varphi}(\zeta^{-1}, t) - \bar{\omega}(\zeta^{-1})\varphi'(\zeta, t)/\omega'(\zeta) + A(t)\bar{\omega}(\zeta^{-1}) + B(t)\omega(\zeta), \quad |\zeta| > 1 \tag{1.7}$$

The function $\varphi(\zeta, t)$ and $\psi(\zeta, t)$ determine the stress–strain state in the domain S and the required loads on its outer boundary L. This solution makes sense if the contour γ of the ζ plane, corresponding to the contour L, lies wholly inside the ring $1 < |\zeta| < R$, in which the functions φ and ψ are holomorphic [1].

Note that, in the case of an elliptic physically non-linear inclusion, the solution for the stress–strain state in the domain S can be continued beyond the boundary L, including an infinitely distant point. In this case, the relations between the stresses σ^∞_{kl} and the rotation ε^∞ at infinity and the analogous quantities in the domain S^* will have the form [1]

$$\begin{aligned}
&(\tilde{\varkappa} + 1)(\sigma^\infty_{11} + \sigma^\infty_{22})/4 + 2i\tilde{\mu}\varepsilon^\infty = A + m_0\bar{B} + \tilde{\mu}(C + m_0 D)\\
&(\tilde{\varkappa} + 1)(\sigma^\infty_{22} - \sigma^\infty_{11} + 2i\sigma^\infty_{12})/2 = -m_0 A - m_0^2\bar{B} + \tilde{\varkappa}(B + m_0 A) -\\
&- \tilde{\mu}(\bar{D} + 2m_0 \operatorname{Re} C + m_0^2 D); \quad m_0 = (a - b)/(a + b)
\end{aligned} \tag{1.8}$$

where a and b are the semi-axes of the ellipse.

2. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

It follows from relations (1.6) and (1.7) and the analogous results for the case of an elastic medium S [1] that the question of the existence of a solution of the problem considered above reduces to the solvability of integral equation (1.6) for the function $\varphi(\zeta, t)$ (subject to the condition that the above-mentioned contour γ lies wholly inside the ring $1 < |\zeta| < R$ on which the right-hand side of Eq. (1.6) is a holomorphic function). We will assume that these known conditions for the solvability of the problem [3] are satisfied.

This, for example, holds if the kernel of the operator K_2 from (1.1) is bounded and the right-hand side of equality (1.6), for any $|\zeta| > 1$, is a summable function of t or if these two functions are summable functions with a square. Operators with a weak singularity such as Abelian kernels [2, 3] with an index λ in the denominator which satisfies the inequality $0 < \lambda < \frac{1}{2}$ belong to the latter case. The sufficient conditions for the unique solvability in the corresponding classes of functions and when $\frac{1}{2} \leq \lambda < 1$ can also be indicated: constraints have been presented [3] for which operators of the potential type (and, in particular, Abelian operators) will be completely continuous when $\lambda \in (0, 1)$.

Using the functions $\varphi(\zeta, t)$ and $\psi(\zeta, t)$ found from (1.6) and (1.7), the required loads on the outer boundary L of the domain S will be uniquely defined [1]. We will now show that a unique stress–strain state in the domain $S^* \cup S$, which has also been obtained above, will correspond to these loads (subject to certain constraints on relations (1.1) and (1.2) which are analogous to those mentioned earlier [1]), that is, the physically non-linear inclusion will exist in a uniform stress–strain state. For this purpose, we write relations (1.1) in a more compact form by expressing $\varepsilon_{kl} = \varepsilon_{kl}(t)$ in terms of $\sigma_{kl} = \sigma_{kl}(t)$.

$$\begin{aligned} &\varepsilon_{kl}(t) = a_{klmn}\sigma_{mn}(t) + \varepsilon^{v}_{kl}(t) \\ &\varepsilon^{v}_{kl}(t) = \int_0^t b_{klmn}(t-\tau)\sigma_{mn}(\tau)d\tau, \quad k, l = 1, 2 \end{aligned} \tag{2.1}$$

where a_{klmn} and b_{klmn} are the components of the elastic pliabilities and the creep kernel which possess well-known symmetry properties.

As earlier in [1], we will now specify constitutive equations (1.2), assuming that their right-hand sides are the sum of the elastic and inelastic ε^{*N}_{kl} strains

$$\varepsilon^{*}_{kl} = a^{*}_{klmn}\sigma^{*}_{mn} + \varepsilon^{*N}_{kl}, \quad k, l = 1, 2 \tag{2.2}$$

The stability conditions for the inelastic strains of the medium S and the inclusion S^*

$$\begin{aligned} &I_1 \geq 0, \quad I_2 \geq 0; \quad I_1(t) \equiv \int_0^t \Delta\dot{\varepsilon}^{v}_{kl}\Delta\sigma_{kl}dt, \quad I_2(t) \equiv \int_0^t \Delta\dot{\varepsilon}^{*N}_{kl}\Delta\sigma^{*}_{kl}dt \\ &\Delta\sigma_{kl}|_{t=0} = \Delta\sigma^{*}_{kl}|_{t=0} = 0, \quad k, l = 1, 2 \end{aligned} \tag{2.3}$$

where Δ is the sign of an increment in the corresponding quantity, will be sufficient for the solution of the problem to be unique.

The first inequality of (2.3) follows from the condition for the work of the stresses in deformations of the linear viscoelastic medium to be positive [2]. Note that for this to be satisfied it is sufficient that the two quadratic forms are positive definite and, in fact,

$$b_{klmn}(x)\xi_{kl}\xi_{mn} > 0, \quad [c_{klmn}(x)]'\xi_{kl}\xi_{mn} > 0 \quad (\xi_{kl}\xi_{kl} \neq 0) \tag{2.4}$$

or any ξ_{kl} ($\xi_{kl} = \xi_{lk}$, $k, l = 1, 2$), where c_{klmn} are the components of the tensor which is the inverse of b_{klmn}; the prime denotes differentiation with respect to x.

Actually, when account is taken of the conditions $\Delta\sigma_{kl}|_{t=0} = 0$, we have

$$I_1(t) = \int_0^t I_3(\tau)d\tau$$

where

$$\begin{aligned} I_3(t) &\equiv x_{kl}(t)\int_0^t b_{klmn}(t-\tau)\dot{x}_{mn}(\tau)d\tau = \\ &= I_4(t) + \frac{1}{2}\int_0^t [c_{klmn}(t-\tau)]' y_{kl}(t,\tau)y_{mn}(t,\tau)d\tau + \frac{1}{2}c_{klmn}(0)y_{kl}(t,t)y_{mn}(t,t) \end{aligned}$$

$$I_4(t)=\int_0^t b_{klmn}(t-\tau)\dot{x}_{mn}(\tau)x_{kl}(\tau)d\tau$$

$$x_{kl}\equiv\Delta\sigma_{kl},\quad y_{kl}(t,\tau)\equiv\int_0^\tau b_{klmn}(t-\xi)\dot{x}_{mn}(\xi)d\xi$$

and, moreover, it has been assumed that $c_{klmn}(0)$ exist.

From this, as a consequence of inequalities (2.4), we have

$$I_1(t)\geqslant I_5(t)\equiv\int_0^t I_4(\xi)d\xi$$

On changing the order of integration in I_5 and integrating by parts, we find that

$$I_5(t)=\frac{1}{2}\int_0^t b_{klmn}(t-\tau)x_{kl}(\tau)x_{mn}(\tau)d\tau$$

regardless of the fact as to whether or not there is a (weak) singularity of the kernels $b_{klmn}(x)$ when $x = 0$.

It therefore follows from inequalities (2.4) that $I_1(t) \geqslant 0$.

Note that known creep kernels of the form [2]

$$b_{klmn}(x)=e^{-\lambda x}\,b^0_{klmn}\quad(\lambda>0)\quad\text{and}\quad b_{klmn}(x)=x^{-\lambda}b^0_{klmn}\quad(0\leqslant\lambda<1)$$
$$b^0_{klmn}=\text{const},\quad b^0_{klmn}\xi_{kl}\xi_{mn}>0\quad\text{when}\quad\xi_{kl}\xi_{lk}\neq 0$$

satisfy conditions (2.4).

The second inequality of (2.3) and the constraints on the constitutive equations of the physically non-linear inclusion of the form (2.2), which follow from it, have been previously investigated in [4].

The proof of uniqueness is analogous to that which was employed previously [1]. Assuming the existence of two solutions which satisfy the same boundary conditions on the contour L (which correspond to the functions $\varphi(\zeta, t)$ and $\psi(\zeta, t)$ found in Section 1), and by applying the virtual work equations to the differences (which we denote using the symbol Δ) of the stresses and rates of deformation, by virtue of the continuity of the loads and displacements in the contour L^*, we will have

$$\int_S \Delta\dot{\varepsilon}_{kl}\Delta\sigma_{kl}dS+\int_{S^*}\Delta\dot{\varepsilon}^*_{kl}\Delta\sigma^*_{kl}dS=0$$

Substituting relations (2.1) and (2.2) into this equality and then integrating with respect to time from zero to the actual instant of time t and assuming that $\Delta\sigma_{kl}\,|_{t=0} = 0$ in the domain S and $\Delta\sigma^*_{kl}\,|_{t=0} = 0$ in the domain S^*, we obtain

$$\int_S\left(u(t)+\int_0^t\Delta\dot{\varepsilon}^{v}_{kl}\Delta\sigma_{kl}dt\right)dS+\int_{S^*}\left(v(t)+\int_0^t\Delta\dot{\varepsilon}^{N*}_{kl}\Delta\sigma^*_{kl}\right)dS=0$$
$$u(t)=\frac{1}{2}a_{klmn}\Delta\sigma_{kl}(t)\Delta\sigma_{mn}(t),\quad v(t)=\frac{1}{2}a^*_{klmn}\Delta\sigma^*_{kl}(t)\Delta\sigma^*_{mn}(t)\tag{2.5}$$

by virtue of the uniqueness of the solution at the corresponding instant of time $t = 0$ of the elastic or elastoplastic problem [1]. By virtue of conditions (2.3), this is only possible when $\Delta\sigma_{kl}(t) = 0$ in the domain S and $\Delta\sigma^*_{kl}(t) = 0$ in the domain S^* for any $t > 0$.

Note that, for the proof of uniqueness, the second inequality of (2.3) can be relaxed considerably (but, like the first inequality, it is however a consequence of the physically based requirement that the work of the stresses in viscoelastic deformations must be positive and, as before, we therefore assume that $I_1 \geqslant 0$): it is sufficient to require that

$$I_2(t)\geqslant-\int_0^t\lambda_0(t)v(t)dt$$

where $\lambda_0(t)$ is a certain continuous positive function. Actually, from relation (2.5) when account is taken of the first inequality of (2.3), we have

$$V(t) \leqslant \int_0^t \lambda_0(\tau)V(\tau)d\tau, \quad V(t) \equiv \int_{S^*} v(t)dS$$

from where, as a result of the well-known Gronwall inequality [5], $V(t) = 0$, that is, $\Delta\sigma_{kl}^*(t) = 0$ in the domain S^*. It then follows from relation (2.5) that $\Delta\sigma_{kl}(t) = 0$ in the domain S.

3. THE OPTIMAL DEFORMATION AND FRACTURE OF A PHYSICALLY NON-LINEAR INCLUSION UNDER CREEP CONDITIONS

We will consider the case when the creep deformations ε_{kl}^{*c} are the inelastic deformations of the inclusion and the constitutive equations for S^* have the form (2.2) when $\varepsilon_{kl}^{*N} \equiv \varepsilon_{kl}^{*c}$, where [6]

$$\begin{aligned} &\eta_{kl}^* \equiv \dot{\varepsilon}_{kl}^{*c} = B_1 s^n (1-\Omega)^{-m} \partial s / \partial \sigma_{kl}^*, \quad k, l = 1, 2 \\ &\dot{\Omega} = B_2 s^p (1-\Omega)^{-m} \end{aligned} \tag{3.1}$$

where $s = s(\sigma_{kl}^*)$ is a homogeneous first-order convex positive function, $\Omega(0 \leqslant \Omega \leqslant 1)$ is a damage parameter (in the natural undeformed state $\Omega = 0$ and at the instant of fracture $\Omega = 1$) and B_1, B_2, m, n, p are positive constants.

Relations (3.1) can be inverted by expressing σ_{kl}^* and $\dot{\Omega}$ in terms of η_{kl}^* and Ω [6]:

$$\begin{aligned} &\sigma_{kl}^* = s\partial H / \partial \eta_{kl}^*, \quad k, l = 1, 2; \quad \dot{\Omega} = B_0 H^\alpha (1-\Omega)^\beta \\ &s = [B_1^{-1} H (1-\Omega)^m]^{1/n}, \quad B_0 = B_2 B_1^{-\alpha}, \quad \alpha = p/n, \quad \beta = m(\alpha - 1) \end{aligned} \tag{3.2}$$

where $H = H(\eta_{kl}^*)$ is a homogeneous first-degree convex positive function such that $Hs = \eta_{kl}^*\sigma_{kl}^*$.

Relations (3.1) (or (3.2)) describe the processes of isothermal deformation under conditions of creep and fracture of materials which weaken in a brittle or viscous manner.

We will now formulate some inverse problems for a viscoelastic medium S with an inclusion S^*, the defining equations for which have the form (1.1), (2.2) and (3.1).

Problem 1 (*on the optimal deformation of an inclusion*). On the outer boundary L of the domain S, those loads p_k, acting over a time interval $[0, t_0]$, have to be selected for which the creep deformations ε_{kl}^{*c} in the domain S^* at the instant of time $t = t_0$ will have the required values ε_{kl**}^{*c}, which are independent of the coordinates of the points of the domain S^*, for the smallest magnitude of the greatest damage $\Omega_{\max} = \max_{S^*} \Omega$ in S^*. Here, the duration t_0 of the external force and the permissible stresses are bounded: $0 < t_0 \leqslant t_{**}$ and $\max_{S^*} s \leqslant s_{**}$ when $0 \leqslant t \leqslant t_0$ (t_{**} and s_{**} are specified quantities).

Problem 2 (*on the optimal fracture of an inclusion*). With the same constraints imposed on t_0 and s, loads p_k on the boundary L are chosen such that the whole domain S^* is fractured after a time $t_0 \leqslant t_{**}$ (that is, in order that $\Omega(x_1, x_2, t_0) = 1$ for all points $(x_1, x_2) \in S^*$) with a minimum level of energy dissipated by creep

$$A^c \equiv \int_{S^*} \int_0^{t_0} \eta_{kl}^* \sigma_{kl}^* dt\, dS$$

In both problems, it is assumed that, when $t < 0$, the whole of the domain $S \cup S^*$ exists in the undeformed state and, hence, $\Omega = 0$ and $\varepsilon_{kl}^{*c} = 0$ $(k, l = 1, 2)$ when $t = 0$ everywhere in S^*.

As previously [6], it is assumed that the above-mentioned constraints are compatible, that is, the corresponding paths for homogeneous deformation exist under the conditions for creep of the domain S^*.

We will now briefly consider each of the problems separately, using results which have been obtained previously [6].

Problem 1. We must distinguish three cases [6], depending on the magnitude of α from (3.2).

1. $\alpha > 1$. Among all paths leading to the specified strains ε^{*c}_{kl**}, the optimal path (in the sense of the accumulation of the least damage) is the path with constant rates of creep strains over the whole possible time interval $[0, t_{**}]$, that is, when $t_0 = t_{**}$ and $\eta^*_{kl} = \varepsilon^{*c}_{kl**}/t_{**}$. The corresponding stresses σ^*_{kl} are determined from the first relation of (3.2)

$$\sigma^*_{kl} = s\frac{\partial H_{**}}{\partial \varepsilon^{*c}_{kl**}}, \quad H_{**} = H(\varepsilon^{*c}_{kl**})$$

$$s = \left(\frac{1}{B_1}\frac{H_{**}}{t_{**}}\right)^{1/n}\left[1-(1-\beta)B_0\left(\frac{H_{**}}{t_{**}}\right)^{\alpha} t\right]^{m/[n(1-\beta)]} \quad (0 \leqslant t \leqslant t_{**}) \tag{3.3}$$

Hence, in this case, the uniform stress–strain state when $0 \leqslant t \leqslant t_{**}$, defined by formulae (2.3) and (3.1)–(3.3), will be the optimal stress–strain state in S^*.

2. $\alpha = 1$. Any simple strain (for the strains ε^{*c}_{kl}) is an optimal strain, that is: when $\varepsilon^{*c}_{kl} = \varepsilon^{*c}_{kl**}f(t)$ ($f(0) = 0$, $\dot{f}(t) > 0$) in any interval $[0, t_0]$ ($t_0 \leqslant t_{**}$) subject to the condition that $s_{\max}(t) \leqslant s_{**}$ ($0 \leqslant t \leqslant t_0$). The stresses σ^*_{kl} are easily found from the first relation of (3.2).

3. $\alpha < 1$. In this case $\sigma^*_{kl}(t) = \text{const}$ when $0 \leqslant t \leqslant t_0$ and $s = s_{**}$, that is, there is a simple strain. From relations (3.1) and (3.2), we find

$$\sigma^*_{kl} = s_{**}\frac{\partial H_{**}}{\partial \varepsilon^{*c}_{kl**}}, \quad t_0 = \frac{1-(1-B_1^{-1}B_2 s_{**}^{p-n}H_{**})^{m+1}}{B_2(m+1)s_{**}^p}$$

which is possible if $t_0 \leqslant t_{**}$.

Hence, in each of the three cases, the corresponding uniform stress–strain state, which is specified by the formulae presented above, must be realized in the domain S^*. The problem therefore reduces to that considered in Section 1, that is, using the known magnitudes of A, B, C and D, the functions $\varphi(\zeta, t)$ and $\psi(\zeta, t)$, which define the stress–strain state in S, and the required loads p_k on L, are found from relations (1.6) and (1.7).

Problem 2. Here, it is also necessary to distinguish three cases, which now depend on the magnitude of $\gamma_0 \equiv p/(n+1)$ [6].

1. $\gamma_0 > 1$. It has been shown in [6] that the strain for which $s(t) = s_{**}$ ($0 < t \leqslant t_0$), where the time t_0 until fracture is determined from relation (3.1): $t_0^{-1} = B_2(m+1)s_{**}^p$ is the optimal strain (in the sense of energy expenditure) for the fracture of an element of the medium existing in a uniform stress–strain state.

2. $\gamma_0 = 1$. In this case, the specific energy expended in fracture $A^c_* = B_1 B_2^{-1}$, that is, it is independent of the strain path.

3. $\gamma_0 < 1$. The strain with a constant specific dissipation power $W \equiv \eta^*_{kl}\sigma^*_{kl} = W_0$ over the course of the maximum possible time, that is, in the interval $[0, t]$, is the optimal strain. In this case [6]

$$W_0 = B_1[B_2(m+1-\gamma_0 m)t_{**}]^{-1/\gamma_0}$$

For the optimal fracture of the physically non-linear inclusion in each of the three cases, it is therefore sufficient to create the corresponding uniform (non-unique) stress–strain state in the domain S^* which satisfies relations (2.2), (3.1) and (3.2), and the above-mentioned equalities for t_0 and s or W. The required functions will be determined from relations (1.6) and (1.7).

We will now formulate a further two problems of optimal fracture which are distinguished from the earlier problems in that the creep strains ε^{*c}_{kl**} at the instant when fracture of the inclusion occurs are specified.

Problems 3a and 3b. With the appropriate choice of loads p_k on the boundary L, it is required to produce in the domain S^* the uniform (solely time-dependent) stress–strain state for which fracture occurs for the specified strains ε^{*c}_{kl**}: (a) with the shortest value of t_0 of the duration of the process and (b) with the least energy dissipation accompanying creep A^c.

The initial conditions are the same as those in Problems 1 and 2.

It is obvious that the problem reduces to finding the optimal strain paths under creep conditions for an element of the medium (for a point) in the case of specified strains ε^{*c}_{kl**} at the instant of fracture with the minimum time or energy expenditures. Note that these problems were not studied in [6].

Before proceeding to consider these problems, we will make some remarks. We consider the set of all strains (or loading) paths of an element of the medium which, at the instant of fracture, leads to the specified strains ε^{*c}_{kl**}. Instead of the variable t, we shall consider the quantity Ω ($0 \leqslant \Omega \leqslant 1$) which, by virtue of relations (3.1) or (3.2), is an increasing function of t, that is, we assume that $\sigma^*_{kl} = \sigma^*_{kl}(\Omega)$ and $\varepsilon^{*c}_{kl} = \varepsilon^{*c}_{kl}(\Omega)$ and that $\varepsilon^{*c}_{kl}(0) = 0$ and $\varepsilon^{*c}_{kl}(1) = \varepsilon^{*c}_{kl**}$ for any path.

Changing in relations (3.1) and (3.2) to differentiation with respect to Ω and taking account of the homogeneity of the function $H = H(\xi_{kl})$ and the equality

$$\frac{d}{d\Omega} = \frac{1}{B_2} s^{-p}(1-\Omega)^m \frac{d}{dt} = \frac{1}{B_0}[H(\eta^*_{kl})]^{-\alpha}(1-\Omega)^{-\beta}\frac{d}{dt}$$

we obtain

$$\varepsilon^{*c'}_{kl} = \frac{B_1}{B_2} s^{n-p}\frac{\partial s}{\partial \sigma^*_{kl}}, \quad H(\varepsilon^{*c'}_{kl}) = \frac{B_1}{B_2}s^{n-p} = \frac{1}{B_0}[H(\eta^*_{kl})]^{1-\alpha}(1-\Omega)^{-\beta} \tag{3.4}$$

where the prime denotes differentiation with respect to Ω.

The time t_* until fracture occurs is found from relations (3.1) and (3.2) and is a functional of $s = s[\sigma^*_{kl}(\Omega)]$ or $H = H[\eta^*_{kl}(\Omega)]$

$$\begin{aligned} t_* &= \frac{1}{B_2}\int_0^1 \{s[\sigma^*_{kl}(\Omega)]\}^{-p}(1-\Omega)^m d\Omega \\ t_* &= \frac{1}{B_0}\int_0^1 \{H[\eta^*_{kl}(\Omega)]\}^{-\alpha}(1-\Omega)^{-\beta} d\Omega \end{aligned} \tag{3.5}$$

In view of the convexity and homogeneity of the function $H = H(\xi_{kl})$, the equality [6]

$$H(\xi^{(2)}_{kl}) \geqslant \partial H/\partial \xi_{kl}\Big|_{\xi_{kl}=\xi^{(1)}_{kl}} \xi^{(2)}_{kl}$$

holds and, from this equality, on putting $\xi^{(1)}_{kl} = \varepsilon^{*c}_{kl}$ and $\xi^{(2)}_{kl} = \varepsilon^{*c'}_{kl}$, we find

$$H(\varepsilon^{*c'}_{kl}) \geqslant [H(\varepsilon^{*c}_{kl})]'$$

where the equality sign only holds for simple strain paths. From this, taking account of relations (3.4), we obtain

$$\begin{aligned} H_{**} &\equiv H(\varepsilon^{*c}_{kl**}) = \int_0^1 [H(\varepsilon^{*c}_{kl})]' d\Omega \leqslant \int_0^1 H(\varepsilon^{*c'}_{kl}) d\Omega = \\ &= \frac{1}{B_0(1-\beta)}\int_0^1 [H(\eta^*_{kl})]^{1-\alpha} d\Omega_1, \quad 1-\Omega_1 = (1-\Omega)^{1-\beta} \quad (1-\beta>0) \end{aligned} \tag{3.6}$$

The condition $\beta < 1$ in (3.6) follows from (3.2) under the assumption that, in the case of strain at a constant rate, that is, when $H = \text{const}$, the time t_* until fracture occurs and, consequently, also the strains at the instant of time $t = t_*$ are finite.

We will now consider each of Problems 3*a* and 3*b* separately.

Problem 3*a*. When $\alpha > 1$, the path with constant strain rates, that is, when $\eta^*_{kl_0} = \varepsilon^{*c}_{kl**}/t_0$, where the time t_0 until fracture occurs is defined as follows:

$$t_0 = [B_0(1-\beta)H^{\alpha}_{**}]^{1/(\alpha-1)} \tag{3.7}$$

is optimal among all the paths from the above-mentioned set and, when $\alpha < 1$, among the simple paths belonging to the same set.

Actually, the equality sign will hold in the case of the (simple) path indicated in (3.6), that is

$$\frac{1}{B_0(1-\beta)}\int_0^1 H_0^{1-\alpha} d\Omega_1 = H_{**}, \quad H_0 \equiv H(\eta^*_{kl_0})$$

Then, for any other path $\eta^*_{kl} = \eta^*_{kl}(\Omega_1)$, we obtain from relations (3.5) and (3.6)

$$t_* = \frac{1}{B_0(1-\beta)}\int_0^1 H^{-\alpha} d\Omega_1, \quad \int_0^1 (H^{1-\alpha} - H_0^{1-\alpha}) d\Omega_1 \geqslant 0, \quad H = H(\eta^*_{kl}) \tag{3.8}$$

Hence, we find

$$t_* - t_0 = \frac{1}{B_0(1-\beta)}\int_0^1 [(H^{1-\alpha})^{\alpha/(\alpha-1)} - (H_0^{1-\alpha})^{\alpha/(\alpha-1)}] d\Omega_1 \geqslant I_6$$

$$I_6 \equiv \frac{1}{B_0(1-\beta)}\frac{\alpha}{\alpha-1}H_0^{-1}\int_0^1 (H^{1-\alpha} - H_0^{1-\alpha}) d\Omega_1 \tag{3.9}$$

where account has been taken of the fact that $H_0 = \text{const}$ and we have used the obvious inequality for the function $f_1(x) = x^{\alpha/(\alpha-1)}$

$$f_1(x) - f_1(x_0) \geqslant f_1'(x_0)(x - x_0)$$

since

$$f_1''(x) = \alpha(\alpha-1)^{-2} x^{(2-\alpha)/(\alpha-1)} > 0$$

when $x > 0$, $\alpha > 0$, $\alpha \neq 1$.

It is clear from relations (3.8) and (3.9) that $I_6 \geqslant 0$ when $\alpha > 1$ for any path and $I_6 = 0$ for any simple path when $\alpha \neq 1$, which also proves the assertion formulated above. Relation (3.7) for the optimal time t_0 follows from (3.8) when $H = H_0 = H_{**}/t_0$.

Note that, under the above-mentioned strain conditions when $H = \text{const}$, the stresses at the instant of fracture are equal to zero, that is, $\sigma^*_{kl}|_{\Omega=1} = 0$ $(k, l = 1, 2)$ which follows from the first relation of (3.2). Hence, according to relation (2.2), we have $\varepsilon^*_{kl}|_{\Omega=1} = \varepsilon^{*c}_{kl**}$, that is, the total deformations at this time are identical to the creep deformations.

In the case when $\alpha = 1$, from the second relation of (3.2), we have $\dot{\Omega} = B_0 H(\eta^*_{kl})$. The equality

$$\int_0^{t_*} H(\eta^*_{kl}) dt = B_0^{-1}$$

holds for any path, and, hence, when $\eta^*_{kl} = \varepsilon^{*c}_{kl**}/t_0$, we obtain $H_{**} = B_0^{-1}$, that is, a solution exists if the specified deformations ε^{*c}_{kl**} satisfy the condition $H(\varepsilon^{*c}_{kl**}) = B_0^{-1}$. It is clear that, when the last condition is satisfied, the path with the largest possible (from physical considerations) value of $H(\eta^*_{kl})$ is the optimal path. For example, if there is a constraint on $s : s(t) \leqslant s_{**}$ (as in Problems 1 and 2), then, according to relations (3.2), $H_{max} = B_1 s^n_{**}(1-\Omega)^{-m}$ and the time up to fracture $t_* = [B_2(m+1)s^n_{**}]^{-1}$.

Problem 3b. When $n + 1 - p > 0$, the path for the case of constant stresses $\sigma^*_{kl} = \sigma^*_{kl_0}$ is the optimal path (at least among all the simple paths). These stresses are determined from relations (3.2) and (3.4)

$$\sigma^*_{kl_0} = s_0 \partial H_{**}/\partial\varepsilon^{*c}_{kl**}, \quad s_0 = (B_1^{-1}B_2H_{**})^{1/(n-p)}$$

To prove this assertion, we note that it follows from relations (3.4) and (3.6) that

$$H_{**} \leqslant \int_0^1 H(\varepsilon^{*c'}_{kl}) d\Omega = \frac{B_1}{B_2}\int_0^1 s^{n-p} d\Omega, \quad A^c = \frac{B_1}{B_2}\int_0^1 s^{n-p+1} d\Omega \tag{3.10}$$

(the equality sign in inequality (3.10) only holds in the case of simple paths).

From inequality (3.1), by analogy with relations (3.8) and (3.9), we find (the zero subscript refers to quantities corresponding to the stresses $\sigma^*_{kl} = \sigma^*_{kl_0}$)

$$\int_0^1 (s^{n-p} - s_0^{n-p}) d\Omega \geqslant 0, \quad A^c - A_0^c = \frac{B_1}{B_2}\int_0^1 [(s^{n-p})^q - (s_0^{n-p})^q] d\Omega \geqslant I_7$$

$$I_7 \equiv \frac{B_1}{B_2} q s_0 \int_0^1 (s^{n-p} - s_0^{n-p}) d\Omega, \quad q = \frac{n-p+1}{n-p} \tag{3.11}$$

The last inequality in (3.11) is satisfied by virtue of the fact that the second derivative of the function $f_2(x) = x^q$ is positive when $n - p + 1 > 0, x > 0, n \neq p$.

It follows from the last inequality in (3.11) that $A^c \geqslant A_0^c$ when $n - p + 1 > 0$ for simple paths ($I_7 = 0$); if, however, $n - p > 0$, then $A^c \geqslant A_0^c$ for any path ($I_7 \geqslant 0$).

As in Problem 3a, the case when $p = n$ (that is, when $\alpha = 1$) requires separate consideration. A solution will exist if $H_{**} = B_0^{-1} = B_1B_2^{-1}$. According to the second relation of (3.10), when $p = n$, the path with the least (physically) possible magnitude of $s = s(\sigma_{kl}^*)$, which, for example, is equal to the non-zero creep limit (if it exists) below which there is practically no creep, is the optimal path.

If there is a constraint on the time t_0 until fracture occurs: $t_0 \leqslant t_{**}$ (as in Problem 2), then, as has been shown in [6] (the case when $\gamma_0 = n(n + 1)^{-1} < 1$ in the problem of optimal fracture), strain with a constant dissipation power W when $t_0 = t_{**}$ will be the optimal strain.

Note that, when $p = n$, Problem 3b corresponds to the problem considered earlier in [6] when $\gamma_0 < 1$ although, in the latter problem, it was not required that

$$\int_0^{t_*} H(\eta_{kl}^*)dt = H_{**} \equiv H(\varepsilon_{kl**}^{*c})$$

(this equality only follows from (3.2) when $\alpha = 1$, that is, when $\gamma_0 = n(n + 1)^{-1}$).

When $n + 1 - p < 0$, the specific energy expended in the fracture is independent of the strain path and is a characteristic of the material [6].

Suppose $n + 1 - p < 0$. In this case, the function $f_2(x) = x^q$ has a negative second derivative and, in the second inequality of (3.11), the sign will change to a positive sign, that is, the path considered above when $s = s_0 =$ const is the most disadvantageous path (at least among all the simple paths) in the sense of the energy expenditure.

The constraint from below $A_0^c \geqslant 0$ is natural in the case of the optimal path (when $n + 1 - p < 0$) and the attainment of the lower boundary $A_0^c \geqslant 0$ is theoretically possible (in the limit). As an example, we consider the two sets of simple paths which depend on the parameters $\xi > 0$

$$\varepsilon_{kl}^{*c} = \varepsilon_{kl**}^{*c}F_i(\Omega), \quad i = 1, 2 \qquad (3.12)$$
$$F_1 = \Omega^\xi, \qquad F_2 = 1 - (1 - \Omega)^\xi$$

In both cases, we find from relations (3.4) and (3.10)

$$A^c = \left(\frac{B_2}{B_1}\right)^{1/(n-p)} H_{**}^q F(\xi), \quad F(\xi) = \frac{\xi^q}{(\xi - 1)q + 1}$$

It can be seen that the function $F \doteq F(\xi)$ has a maximum when $\xi = 1$ (which, according to relations (3.4) and (3.12), corresponds to the above-mentioned case when $s =$ const) and $F(\xi) \to 0$ when $\xi \to 0$ and $\xi \to \infty$. It is obvious that these limiting situations lose their physical meaning since, as a consequence of relations (3.12), they lead to the instantaneous strain from $H = 0$ when $\Omega = 0$ to $H = H_{**}$ when $0 < \Omega \leqslant 1$ or from $H = 0$ when $0 \leqslant \Omega < 1$ to $H = H_{**}$ when $\Omega = 1$. It is therefore only possible to consider fairly small (or large) values of ξ for which the stresses and the time until fracture occurs will be finite quantities which must also hold in any actual process.

An analysis of relations (3.12) using equalities (3.1) and (3.4) shows that these constraints will be satisfied if $\alpha^{-1} < \xi < 1$ for $F_1(\xi)$ and $\alpha^{-1}(1 - \beta) < \xi < 1$ for $F_2(\xi)$, where $\alpha = p/n > 1 + n^{-1}$, $\beta = m(\alpha - 1) > m/n$ and $\beta < 1$ (which has been mentioned above). It can be seen from this that the lower boundary for ξ in the case of F_1 lies more to the right than in the case of F_2. Hence, $F_2 = F_2(\xi)$ can be chosen as the function which characterizes a strain path close to the optimal path and which takes account of constraints of a physical nature if the value of ξ is close to $\alpha^{-1}(1 - \beta)$. The corresponding stresses and the time until fracture occurs are easily found from relations (3.2), (3.4) and (3.5).

4. EXAMPLES

Consider an infinite viscoelastic domain (plane) S containing an elliptic physically non-linear inclusion, the constitutive equations for which have the form (2.2) and (3.1). By analogy with (2.1), we solve relations (1.1) for S with respect to the strains, assuming for simplicity that the creep kernels acting on the spherical and deviator parts of the planar stress tensor only differ by constant factors

$$\varepsilon_{kl} = \frac{\tilde{\varkappa}-1}{8\tilde{\mu}}\sigma_{nn}\delta_{kl} + \frac{1}{2\tilde{\mu}}\sigma_{kl}^0, \quad k, l = 1, 2 \tag{4.1}$$

Here,

$$\frac{\tilde{\varkappa}-1}{\tilde{\mu}}x = \frac{\varkappa-1}{\mu}J_1(x), \quad \frac{1}{\tilde{\mu}}x = \frac{1}{\mu}J_2(x)$$

$$J_k(x) \equiv x(t) + \beta_k \int_0^t K(t-\tau)x(\tau)d\tau, \quad \beta_k = \text{const}, \quad k = 1, 2$$

Then, according to relations (1.8) and (4.1), for a specified uniform stress–strain state in the domain S^*, the stresses and rotation at infinity will be determined from the system of equations

$$\frac{\varkappa+1}{\mu}\left[X_k(t) - \beta_0 \int_0^t K(t-\tau)X_k(\tau)d\tau\right] = \Phi_k, \quad k = 1, 2 \tag{4.2}$$

$$X_1 = \frac{\sigma_{11}^\infty + \sigma_{22}^\infty}{4} + 2i\frac{\tilde{\mu}}{\tilde{\varkappa}+1}\varepsilon^\infty, \quad X_2 = \frac{\sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty}{2}$$

$$\Phi_1 = C + m_0 D + \frac{1}{\tilde{\mu}}(A + m_0\overline{B})$$

$$\Phi_2 = -(\overline{D} + m_0^2 D + 2m_0 \operatorname{Re} C) + \frac{\tilde{\varkappa}}{\tilde{\mu}}(B + m_0 A) - \frac{m_0}{\tilde{\mu}}(A + m_0\overline{B})$$

$$\beta_0 = -\frac{\beta_1(\varkappa-1) + 2\beta_2}{\varkappa+1}$$

It is well known [2, 3] that the solution of Eqs (4.2) has the form

$$X_k(t) = \frac{\mu}{\varkappa+1}\left[\Phi_k(t) + \beta_0 \int_0^t \Gamma(\beta_0, t-\tau)\Phi_k(\tau)d\tau\right], \quad k = 1, 2 \tag{4.3}$$

where $\Gamma(\beta_0, t-\tau)$ is the resolvent of the kernel $K(t-\tau)$ from (4.2). For example, in the case of the Abelian kernel which is very common in the theory of linear viscoelastic media, the thoroughly studied Rabotnov fractional-exponential function [2] is the resolvent.

Since an ellipse with semi-axes a and b serves as the boundary L^*, the transformation function has the form [1]

$$\omega(\zeta) = R_0(\zeta + m_0\zeta^{-1}), \quad 2R_0 = a + b, \quad 2m_0R_0 = a - b$$

In the case of a uniform stress–strain state in the domain S^*, the points of the contour L^*, which correspond to the values $z = z_0 \equiv \omega(\sigma)$ ($\sigma = e^{i\theta}$), transfer, after deformation, to a new position, which is determined by the relation $z = z_0 + w^*(z_0, \bar{z}_0)$ whence, as a consequence of relations (4.1), we obtain

$$z = (1 + C/2)\omega(\sigma) + (D/2)\overline{\omega(\sigma)} = R_1 e^{i\alpha_1}(\sigma_1 + m_1\sigma_1^{-1}) \tag{4.4}$$

$$\sigma_1 = e^{i\theta_1}, \quad \theta_1 = \theta + (\varphi_1 - \varphi_2)/2, \quad \alpha_1 = (\varphi_1 + \varphi_2)/2$$

$$R_1 e^{i\varphi_1} = R_0[1 + (C + m_0D)/2], \quad m_1R_1e^{i\varphi_2} = R_0[m_0 + (m_0C + D)/2]$$

Hence, the contour L^* will be deformed into a new ellipse with semi-axes $R_1(1 + m_1)$ and $R_1(1 - m_1)$ (these quantities are positive in view of the smallness of the strains ε_{kl}^* and the rotation ε^*). Its axes of symmetry are rotated relative to the old axes by an angle α_1. Consequently, by a choice of the functions $C(t)$ and $D(t)$ from (1.4), it is possible to deform the initial boundary L^* into the required ellipse (which, naturally, is close in shape to the initial ellipse) with the parameters $R_1(t)$, $m_1(t)$ and $\alpha_1(t)$ ($0 \leqslant t \leqslant t_*$, t_* is the instant of fracture).

Suppose, for example, that, when $t = t_*$, the initial contour $L^*: z = R_0(\sigma + m_0\sigma^{-1})$ ($0 < m_0 \ll 1$) must be converted into a circle of radius R_1. Then, from relation (4.4), on putting $\alpha_1 = 0$, we obtain for C_* and D_* (magnitudes at the instant of time $t = t_*$ are denoted by an asterisk)

$$\begin{aligned} &C_* + m_0 D_* = 2(R_1 R_0^{-1} e^{i\varphi_{1*}} - 1) \\ &m_0 C_* + D_* = -2m_0 \end{aligned} \tag{4.5}$$

Suppose the rotation $\varepsilon_*^* = 0$ (or $\varepsilon_{12*}^* = 0$). Then (since m_0 is a real constant), it follows from (4.5) and (1.4) that $\varepsilon_{12*}^* = 0$ (or, correspondingly, $\varepsilon_*^* = 0$), $\varphi_{1*} = 0$ and

$$\varepsilon_{kk*}^* = \frac{R_1 R_0^{-1} - (1 + (-1)^{k+1} m_0)}{1 - (-1)^{k+1} m_0}, \quad k = 1, 2 \tag{4.6}$$

If, on the other hand, an initial circular contour $L^*: z = R_0\sigma$ is converted into the ellipse $z = R_1(\sigma + m_1\sigma^{-1})$, then, from relations (4.4) when $\varphi_1 = \varphi_2 = m_0 = 0$, we find

$$\varepsilon_{12*}^* = \varepsilon_*^* = 0, \quad \varepsilon_{kk*}^* = (1 + (-1)^{k+1} m_1) R_1 R_0^{-1} - 1, \quad k = 1, 2 \tag{4.7}$$

If the path with the constant rates of creep strains (as, for example, in Problem 3*a*) is the optimal path, then, on taking account of the fact mentioned above that, under such conditions, the total strains and the creep strains are identical at the instant when fracture occurs, we find that the equalities $\eta_{kl0}^* = \varepsilon_{kl*}^*/t_0$ are satisfied for the path in questions, where the magnitude of t_0 is given by expression (3.7). The corresponding stresses σ_{kl}^* are found from relation (3.2) and the required stresses σ_{kl}^∞ are found from (4.3).

In the two examples considered, the strains ε_{kl*}^* are given by relations (4.6) and (4.7), during which $\sigma_{12*}^* = 0$, that is, $\eta_{120}^* = 0$. It follows from this (at least, in the case of an isotropic medium) that $\sigma_{12}^* = 0$ and, therefore, that $\varepsilon_{12}^* = 0$ at any instant of time $t \in [0, t_0]$. Then, the magnitudes of the rotations in the domain S^* and at infinity will be equal, since the condition for the imaginary parts in the first equation of (4.2) to be equal has the form

$$\varepsilon^\infty = \varepsilon^* + m_0\left(\varepsilon_{12}^* - \frac{1}{2\mu}\sigma_{12}^*\right)$$

Hence $\varepsilon^\infty = 0$, if it is assumed that $\varepsilon^* = 0$ for any $t \in [0, t_0]$.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00551).

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Translated by E.L.S.